Orientations, lattice polytopes, and group arrangements III: Cartesian product arrangements and applications to the Tutte type polynomials of graphs

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Abstract. A common generalization for the chromatic polynomial and the flow polynomial of a graph G is the Tutte polynomial $T(G;x,y)$. The combinatorial meaning for the coefficients of T was discovered by Tutte at the beginning of its definition. However, for a long time the combinatorial meaning for the values of T is missing, except for a few values such as $T(G;i,j)$, where $1 \leq i,j \leq 2$, until recently for $T(G;1,0)$ and $T(G;0,1)$. In this third one of a series of papers, we introduce product valuations, cartesian product arrangements, and multivariable characteristic polynomials, and apply the theory of product arrangement to the tension-flow group associated with graphs. Three types of tension-flows are studied in details: elliptic, parabolic, and hyperbolic; each type produces a two-variable polynomial for graphs. Weighted polynomials are introduced and their reciprocity laws are obtained. The dual versions for the parabolic case turns out to include Whitney's rank generating polynomial and the Tutte polynomial as special cases. The product arrangement part is of interest for its own right. The application part to graphs can be modified to matroids.

1. Introduction

The Tutte polynomial $T(G;x,y)$ of a graph G is of fundamental importance in graph theory, for it is a common generalization of the chromatic polynomial $\chi(G,t)$ and the flow polynomial $\varphi(G,t)$, for it has enumerative applications in combinatorics, and contains invariants as specializations of polynomials in knots and partition functions in statistical physics; see [**7, 8, 14, 28, 29**]. It is well-known that the chromatic polynomial χ is coincidentally equal to the characteristic polynomial of a hyperplane arrangement associated with the graph G (called the graphical arrangement in [**23, 25**]). Analogously, the flow polynomial φ is coincidentally equal to the characteristic polynomial of the flow arrangement associated with G; see [**1, 14, 17, 25**]. It is then natural to ask whether the Tutte polynomial

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T appears as ceratin two-variable characteristic polynomial of subspace arrangement associated with G. Based on the work in the first two papers of the series [**8, 14**], we address in this third one the issues that have been studied for the tension polynomial (equivalent to the chromatic polynomial) and the flow polynomial to the Tutte polynomial of graphs.

Notice that the chromatic polynomial arises from the pattern of the number of proper colorations of a graph with a given set of colors in terms of the cardinality of the color set, regardless of the internal relations of the colors. More specifically, given a color set A; the set of colorations of any set S by A is the set A^S of all functions from S to A, the set $C_{\rm nz}(G,A)$ of proper colorations of G is just the set of all proper functions from the vertex set of G to A. The configuration set $C_{\rm nz}$ (precisely its indicator) can be expressed by inclusion-exclusion as a linear expression in terms of A^S (precisely their indicators) for some vertex subsets S. If A is finite, the cardinalities $|A^S|$ for various S have the patterns $|A|^0, |A|^1, |A|^2$, etc.; subsequently, the cardinality $|C_{\rm nz}|$ has a polynomial pattern in terms of the cardinality $|A|$; such a polynomial is known as the chromatic polynomial of G; see [**3**]. If A is infinite, counting the number of elements of $C_{\rm nz}$ does not make sense, however, the patterns are still there. In fact, when A is an abelian group, say, A is the ring $\mathbb{Z}$ of integers or the field $\mathbb{R}$ of real numbers, the sets A^S have the patterns $[A]^0, [A]^1, [A]^2$, etc., and $C_{\rm nz}$ has a polynomial pattern in terms of $[A]$; such a polynomial is exactly the same as the chromatic polynomial as they have the same pattern.

The situation for the flow polynomial is analogous, provided that A is required to be an abelian group in order to make the conservation equations meaningful at each vertex. More specifically, given an orientation ε on the graph G and an abelian group A, the flow group $F(G,\varepsilon;A)$ of all flows on the digraph (G,ε) has a subgroup arrangement, known as *flow arrangement*, consisting of flow subgroups F_e of flows vanishing on an edge e, where e ranges over all edges; see [**25**]. The flow polynomial $\varphi(G,t)$ is then the characteristic polynomial of the flow arrangement. The philosophy may apply to other polynomials or functions arising from "counting" in combinatorics and other fields.

Back to the problem of expressing the Tutte polynomial $T(G;x,y)$ as possible two-variable characteristic polynomial of certain group arrangement associated with the graph G, one has to require the objects in the arrangement to be certain products to produce two variables. Where the products come from naturally? It is not random to consider the cartesian product of the coloration group $C(G,A)$ of all colorations of G by a color group A and the flow group $F(G,\varepsilon;B)$ over an abelian group B. Notice that $C(G,A)$ can be naturally transformed into the tension group $T(G,\varepsilon;A)$ by the difference operator (see Appendix 2); so it is natural to work within the tension-flow group $\Omega := T(G,\varepsilon;A)\times F(G,\varepsilon;B)$. Of course we shall not consider all tension-flows in Ω. The tension-flows $(f,g)\in\Omega$ that we have interests are the following three types: (i) $\operatorname{supp} f\subseteq \ker g$, called *elliptic*; (ii) $\operatorname{supp} f=\ker g$, called *parabolic*; and (iii) $\ker f\subseteq \operatorname{supp} g$, called *hyperbolic*.

We shall see that the counting of these tension-flows yields two-variable polynomials when an appropriate valuation (finitely additive measure) is equipped on Ω, provided that A,B are finite, or finitely generated abelian groups, or infinite fields. For the parabolic (also referred to complementary) case, there are dual polynomials,

which are exactly the existing Whitney polynomial $R(G;x,y)$ or the Tutte polynomial $T(G;x+1,y+1)$. Our approach automatically gives rise to combinatorial and geometric interpretations for $T(G;x,y)$.

The paper is arranged as follows. Cartesian product arrangements are introduced in Section 2 and a foundation for product valuations is set up in general. The multivariable characteristic polynomial is introduced as the total measure of a unique valuation of the complement for any product arrangement. In Section 3, a two-variable characteristic polynomial is produced by a product arrangement on Ω by considering nowhere-zero tension-flows; this polynomial corresponds to the hyperbolic case of tension-flows. In Section 4, weighted integral complementary polynomial is introduced and its dual is obtained. These weighted polynomials contain four variables, generalizing the integral complementary polynomial in [**13**] and making a true reciprocity law for such polynomials. The modular case of weighted complementary tension-flows are studied in Section 5. Other weighted counting of tension-flows are considered in Section 6. The product arrangement part is of interest for its own right. The application part to graphs can be modified to matroids. The first version of the paper was finished in 2007 and was reported in the 2007 International Conference on Graphs and Combinatorics; see [**12**].

2. Cartesian product arrangements

Let S be a non-empty set. A collection $\mathcal{L}$ of subsets of S is called an *intersectional class* if any finite intersection[1] of sets from $\mathcal{L}$ is also a member of $\mathcal{L}$. A class of subsets of S is called a (*relative*) *Boolean algebra* if it is closed under finite intersection, union, and (relative) complement. For any class $\mathcal{A}$ of subsets of S, we denote by $\mathscr{L}(\mathcal{A})$ the smallest intersectional class that contains $\mathcal{A}$. Then $\mathscr{L}(\mathcal{A})$ consists of all possible finite intersections of sets from $\mathcal{A}$, called the *semilattice* generated by $\mathcal{A}$. For an intersectional class $\mathcal{L}$ of S, we denote by $\mathscr{B}(\mathcal{L})$ the smallest relative Boolean algebra that contains $\mathcal{L}$, and say that $\mathscr{B}(\mathcal{L})$ is generated by $\mathcal{L}$, and is further generated by a class $\mathcal{A}$ of subsets if $\mathcal{L}$ is the semilattice $\mathscr{L}(\mathcal{A})$.

Let Ω be the cartesian product $\prod_{i=1}^n \Omega_i$ of non-empty sets Ω_i. Let $\mathscr{L}_i$ be an intersectional class of Ω_i, and let $\mathscr{B}_i$ be the relative Boolean algebra generated by $\mathscr{L}_i$. We denote by $\prod_{i=1}^n \mathscr{L}_i$ the smallest intersectional class of Ω that contains the products $\prod_{i=1}^n A_i$, where $A_i \in \mathscr{L}_i$; and denote by $\prod_{i=1}^n \mathscr{B}_i$ the relative Boolean algebra generated by $\prod_{i=1}^n \mathscr{L}_i$.

Proposition 2.1. *Given a valuation ν_i on each $\mathscr{B}_i$ with values in a commutative ring R, where $1 \le i \le n$. There exits a unique valuation $\nu : \prod_{i=1}^n \mathscr{B}_i \to R$ such that for each B_i of $\mathscr{B}_i$,*

$$\nu\left(\prod_{i=1}^n B_i\right) = \prod_{i=1}^n \nu_i(B_i).$$

Proof. Let X be an object of $\prod_{i=1}^n \mathscr{B}_i$ and be written in finite disjoint union of the form $X = \bigsqcup_{(j_i)} \prod_{i=1}^n A_{i,j_i}$, where $A_{i,j_i} \in \mathscr{B}_i$. We define

$$\nu(X) := \sum_{(j_i)} \nu\left(\prod_{i=1}^n A_{i,j_i}\right) = \sum_{(j_i)} \prod_{i=1}^n \nu_i(A_{i,j_i}).$$

[1] By convention the intersection of none of sets is assumed to be the whole set S; such an intersection is not considered to be a finite intersection unless it is stated otherwise.

It suffices to show that ν is well-defined. Let X be written in another form $X = \bigsqcup_{(k_i)} \prod_{i=1}^n B_{i,k_i}$, where $B_{i,k_i} \in \mathscr{B}_i$. For each $1 \leq i \leq n$, we may refine the collection $\mathscr{C}_i = \{A_{i,j_i}, B_{i,k_i}, A_{i,j_i} \cap B_{i,j_i} \mid j_i, k_i\}$ into a sub-collection $\mathscr{D}_i$ of $\mathscr{B}_i$, consisting of disjoint subsets so that each member in $\mathscr{C}_i$ is a union of some members in $\mathscr{D}_i$. Then

$$\prod_{i=1}^n A_{i,j_i} = \bigsqcup_{\substack{D_i \in \mathscr{D}_i, D_i \subseteq A_{i,j_i} \\ i=1,2,\ldots,n}} \prod_{i=1}^n D_i,$$

$$\prod_{i=1}^n B_{i,k_i} = \bigsqcup_{\substack{D_i \in \mathscr{D}_i, D_i \subseteq B_{i,k_i} \\ i=1,2,\ldots,n}} \prod_{i=1}^n D_i.$$

By definition of ν, we have

$$\nu\Big(\prod_{i=1}^n A_{i,j_i}\Big) = \sum_{\substack{D_i \in \mathscr{D}_i, D_i \subseteq A_{i,j_i} \\ i=1,2,\ldots,n}} \prod_{i=1}^n \nu_i(D_i),$$

$$\nu\Big(\prod_{i=1}^n B_{i,k_i}\Big) = \sum_{\substack{D_i \in \mathscr{D}_i, D_i \subseteq B_{i,k_i} \\ i=1,2,\ldots,n}} \prod_{i=1}^n \nu_i(D_i).$$

Note that $\prod_{i=1}^n D_i \subseteq X$ is equivalent to $\prod_{i=1}^n D_i \subseteq \prod_{i=1}^n A_{i,j_i}$ for some (j_i), and is also equivalent to $\prod_{i=1}^n D_i \subseteq \prod_{i=1}^n B_{i,k_i}$ for some (k_i). Thus

$$\begin{aligned}
\sum_{(j_i)} \prod_{i=1}^n \nu_i(A_{i,j_i}) &= \sum_{(j_i)} \sum_{\substack{D_i \in \mathscr{D}_i, D_i \subseteq A_{i,j_i} \\ i=1,2,\ldots,n}} \prod_{i=1}^n \nu_i(D_i) \\
&= \sum_{\substack{D_i \in \mathscr{D}_i, 1 \leq i \leq n \\ D_1 \times \cdots \times D_n \subseteq X}} \prod_{i=1}^n \nu_i(D_i) \\
&= \sum_{(k_i)} \prod_{i=1}^n \nu_i(B_{i,k_i}).
\end{aligned}$$

This means that $\nu(X)$ is well-defined. □

Let $\Omega = \prod_{i=1}^n \Omega_i$ be the product group of abelian groups Ω_i, either all are finitely generated or all are vector spaces over an infinite field $\Bbbk$. Let $\mathscr{L}(\Omega_i)$ be the intersectional class generated by cosets of all subgroups of Ω_i; let $\mathscr{B}(\Omega_i)$ be the Boolean algebra generated by $\mathscr{L}(\Omega_i)$. Then $\prod_{i=1}^n \mathscr{L}_i(\Omega_i)$ is an intersectional class consisting of all products $\prod_{i=1}^n A_i$, where A_i are cosets of some subgroups of Ω_i; and $\prod_{i=1}^n \mathscr{B}(\Omega_i)$ is the Boolean algebra generated by $\prod_{i=1}^n \mathscr{L}(\Omega_i)$. For each abelian group Γ, either finitely generated or a vector space over an infinite field, the *size* of Γ is defined as

$$|\Gamma| := |\mathrm{Tor}(\Gamma)|\, t^{\mathrm{rank}\,(\Gamma)},$$

where $\mathrm{Tor}(\Gamma)$ is the torsion subgroup of Γ whose elements have finite orders, and $|\mathrm{Tor}(\Gamma)|$ is the cardinality of $\mathrm{Tor}(\Gamma)$. When Γ is a vector space, then $\mathrm{Tor}(\Gamma) = \{0\}$ and the rank is meant the dimension.

Theorem 2.2. *Let $\Omega = \prod_{i=1}^{n} \Omega_i$ be the product group of abelian groups Ω_i, either all are finitely generated or all are vector spaces over an infinite field $\Bbbk$. Then there exists a unique translation-invariant valuation*

$$\lambda : \prod_{i=1}^{n} \mathscr{B}(\Omega_i) \to \mathbb{Q}[t_1, \ldots, t_n]$$

such that for subgroups A_i of Ω_i,

$$\lambda\Bigg(\prod_{i=1}^{n} A_i \Bigg) = \prod_{i=1}^{n} \frac{|\mathrm{Tor}(\Omega_i)|}{|\mathrm{Tor}(\Omega_i/A_i)|}\, t_i^{\mathrm{rank}(A_i)}. \tag{2.1}$$

Proof. Let $\lambda_i : \mathscr{B}(\Omega_i) \to \mathbb{Q}[t_i]$ be the unique translation-invariant valuation such that for each subgroup A_i of Ω_i,

$$\lambda_i(A_i) = \frac{|\mathrm{Tor}(\Omega_i)|}{|\mathrm{Tor}(\Omega_i/A_i)|} t_i^{\mathrm{rank}\,(A_i)};$$

see [**8**] (p.429-433) for the case of finitely generated abelian groups and [**16**] for the case of vector spaces. Then the product $\lambda := \prod_{i=1}^{n} \lambda_i$ is a translation-invariant valuation satisfying (2.1). The uniqueness follows immediately from the uniqueness of λ_i. $\square$

Let $\pi_i : \Omega \to \Omega_i$ be the obvious projection. A *cartesian product arrangement* $\mathcal{A}$ of Ω is a finite collection of cartesian products $\prod_{i=1}^{n} F_i$, where F_i are cosets of some subgroups of Ω_i. The *semilattice* of $\mathcal{A}$ is the collection $L(\mathcal{A})$ of all possible non-empty intersections of sets from $\mathcal{A}$, including the intersection of none of sets, which is assumed to be the whole group Ω. Let μ be the Möbius function of the poset $L(\mathcal{A})$, whose partial order is the set-inclusion. We introduce the *multivariable characteristic polynomial*

$$\chi(\mathcal{A}; t_1, \ldots, t_n) := \sum_{X \in L(\mathcal{A})} \mu(X, \Omega) \prod_{i=1}^{n} \frac{|\mathrm{Tor}(\Omega_i)|}{|\mathrm{Tor}\big(\Omega_i/\pi_i\langle X\rangle\big)|}\, t_i^{\mathrm{rank}\,\pi_i\langle X\rangle}, \tag{2.2}$$

which has integer coefficients whenever Ω_i are vector spaces. This generalizes the one-variable characteristic polynomial [**25, 30**], and will be useful to unify some multivariable polynomials arising in combinatorics such as the Tutte polynomial of graphs and matroids. For instance, the main theorem of the book by Crapo and Rota [**15**] can be obtained by the characteristic polynomial of cartesian product arrangement.

Theorem 2.3. *Let $\mathcal{A}$ be a cartesian product arrangement of the product space $\Omega := \prod_{i=1}^{n} \Omega_i$ of abelian groups Ω_i, either all are finitely generated or all are vector spaces over an infinite field $\Bbbk$. Then*

$$\chi\big(\mathcal{A}; t_1, \ldots, t_n\big) = \lambda\Bigg(\Omega - \bigcup_{A \in \mathcal{A}} A\Bigg). \tag{2.3}$$

Proof. The idea is similar to that of [**9**]. For each subset $S \subseteq \Omega$, let 1_S denote the indicator function of S, i.e., $1_S(x) = 1$ for $x \in S$ and $1_S(x) = 0$ for $x \in \Omega - S$. For each member X of $L(\mathcal{A})$, define the set

$$X^0 := X - \bigcup_{Z \in L(\mathcal{A}), Z < X} Z.$$

Then $\{X^0 \mid X \in L(\mathcal{A})\}$ is a collection of disjoint subsets of Ω. Moreover, each member Y of $L(\mathcal{A})$ can be written as a disjoint union $Y = \bigsqcup_{X\in L(\mathcal{A}),\, X\le Y} X^0$, so that

$$1_Y = \sum_{X\in L(\mathcal{A}),\, X\le Y} 1_{X^0}.$$

By the Möbius inversion,

$$1_{Y^0} = \sum_{X\in L(\mathcal{A}),\, X\le Y} \mu(X,Y)\, 1_X, \quad Y \in L(\mathcal{A}).$$

In particular, since $\Omega^0 = \Omega - \bigcup_{A\in\mathcal{A}} A$, we have

$$1_{\Omega^0} = \sum_{X\in L(\mathcal{A})} \mu(X,\Omega)\, 1_X. \tag{2.4}$$

By Groemer's extension theorem [**18**], any valuation on a relative Boolean algebra of sets can be uniquely extended to a valuation (or integral) on the vector space generated by the indicator functions of sets from the given Boolean algebra. Applying the valuation λ to both sides of (2.4), we obtain

$$\lambda\Big(\Omega - \bigcup_{A\in\mathcal{A}} A\Big) = \sum_{X\in L(\mathcal{A})} \mu(X,\Omega)\lambda(X). \tag{2.5}$$

Since X is a product of cosets of some subgroups of Ω_i, i.e., $X = \prod_{i=1}^n \pi_i(X)$, we see that

$$\begin{aligned}\lambda(X) &= \prod_{i=1}^n \lambda_i(\pi_i(X)) = \prod_{i=1}^n \lambda_i(\pi_i\langle X\rangle) \\ &= \prod_{i=1}^n \frac{|\mathrm{Tor}(\Omega_i)|}{|\mathrm{Tor}(\Omega_i/\pi_i\langle X\rangle)|}\, t_i^{\mathrm{rank}\,\pi_i\langle X\rangle}.\end{aligned}$$

Substitute $\lambda(X)$ into (2.5); we obtain the formula (2.3). □

3. The tension-flow arrangement

Let A, B be abelian groups. A *tension-flow* of (G,ε) is an element (f,g) in the *tension-flow group*

$$\Omega = \Omega\big(G,\varepsilon; A,B\big) := T(G,\varepsilon;A)\times F(G,\varepsilon;B). \tag{3.1}$$

The pair (f,g) can be viewed as a function from E to the abelian group $A\times B$. A tension-flow (f,g) is said to be *nowhere-zero* if

$$(f(e),g(e)) \neq (0,0) \quad \text{for all edges } e\in E.$$

Let $\Omega_{\mathrm{nz}} = \Omega_{\mathrm{nz}}(G,\varepsilon;A,B)$ denote the set of all nowhere-zero tension-flows of (G,ε). Whenever $A = B$, we simply write $\Omega(G,\varepsilon;A)$ for $\Omega(G,\varepsilon;A,B)$, $\Omega_{\mathrm{nz}}(G,\varepsilon;A)$ for $\Omega_{\mathrm{nz}}(G,\varepsilon;A,B)$; and whenever $A=\mathbb{R}$, we further write $\Omega(G,\varepsilon)$ for $\Omega(G,\varepsilon;\mathbb{R})$, $\Omega_{\mathrm{nz}}(G,\varepsilon)$ for $\Omega_{\mathrm{nz}}(G,\varepsilon;\mathbb{R})$. For subsets $X,Y\subseteq E$, let $T_X(G,\varepsilon,A)$ denote the tension subgroup consisting of those tensions vanishing on X, $F_Y(G,\varepsilon,B)$ the flow subgroup consisting of those flows vanishing on Y, and define the tension-flow subgroup

$$\Omega_{X,Y} = \Omega_{X,Y}(G,\varepsilon;A,B) := T_X(G,\varepsilon;A)\times F_Y(G,\varepsilon;B). \tag{3.2}$$

If $X = Y$, we simply write Ω_X for $\Omega_{X,X}$.

Let $|A| = p$ and $|B| = q$ be finite. We introduce the counting function

$$\omega(G;p,q) := |\Omega_{\rm nz}(G,\varepsilon;A,B)|, \tag{3.3}$$

which is a polynomial function of positive integers p, q, and is independent of the chosen orientation ε and the group structures of A, B, called the *hyperbolic tension-flow polynomial* of G. It is called hyperbolic because (f,g) is allowed to have nonzero values at an edge for both f and g. Notice that for subsets $X, Y \subseteq E$,

$$|\Omega_{X,Y}(G,\varepsilon;A,B)| = p^{r\langle E\rangle - r\langle X\rangle} q^{n\langle Y^c\rangle}, \tag{3.4}$$

where $\langle X\rangle = (V,X)$ is the spanning subgraph with the edge set X and $Y^c := E - Y$.

The *tension-flow arrangement* of (G,ε) is a subgroup arrangement $\mathcal{A}(G,\varepsilon;A,B)$ of the tension-flow group $\Omega(G,\varepsilon;A,B)$, consisting of the subgroups

$$\Omega_e := \{(f,g) \in \Omega(G,\varepsilon;A,B) \mid (f,g)(e) = 0\}, \quad e \in E. \tag{3.5}$$

It is clear that $\Omega_e = T_e \times F_e$, where $T_e = T_{\{e\}}$ and $F_e = F_{\{e\}}$. So $\mathcal{A}$ is a cartesian subgroup arrangement of Ω; its semilattice $L(\mathcal{A})$ consists of the subgroups Ω_X, where $X \subseteq E$. The complement of $\mathcal{A}$ is

$$\Omega_{\rm nz} = \Omega - \bigcup_{e\in E} \Omega_e. \tag{3.6}$$

To see that $\omega(G;p,q)$ is independent of the chosen orientation ε, the involution $P_{\varrho,\varepsilon} : A^E \to A^E$, defined for each orientation ϱ of G by

$$(P_{\varrho,\varepsilon} f)(e) = [\varrho,\varepsilon](e) f(e), \quad f \in A^E, e \in E, \tag{3.7}$$

is a group isomorphism from $\Omega(G,\varepsilon;A,B)$ to $\Omega(G,\varrho;A,B)$.

Theorem 3.1. *Let λ be the unique product valuation on $\Omega(G,\varepsilon;\Bbbk)$ with values in $\mathbb{Q}[x,y]$, where $\Bbbk \in \{\mathbb{Z},\mathbb{Q},\mathbb{R},\mathbb{C}\}$. Then for positive integers p, q,*

$$\omega(G;p,q) = \chi(\mathcal{A}(G);p,q) = \lambda(\Omega_{\rm nz}(G))\big|_{(x,y)=(p,q)}. \tag{3.8}$$

Moreover, the polynomial ω has the expansion

$$\omega(G;x,y) = \sum_{X\subseteq E} (-1)^{|X|} x^{r\langle E\rangle - r\langle X\rangle} y^{n\langle X^c\rangle}. \tag{3.9}$$

Proof. It follows from Theorem 2.3 that $\chi(\mathcal{A}(G);x,y) = \lambda(\Omega_{\rm nz}(G))$. Recall (3.6) and apply inclusion-exclusion; we have

$$1_{\Omega_{\rm nz}} = 1_{\Omega - \bigcup_{e\in E} T_e\times F_e} = \sum_{X\subseteq E} (-1)^{|X|}\, 1_{T_X\times F_X}. \tag{3.10}$$

Note that for each subset $X \subseteq E$, if $|A| = p$ and $|B| = q$, then

$$|T_X(G,\varepsilon;A) \times F_X(G,\varepsilon;B)| = p^{r\langle E\rangle - r\langle X\rangle} q^{n\langle X^c\rangle}.$$

Applying the valuation λ and the counting measure to both sides of (3.10), the formulas (3.8) and (3.9) follow immediately. □

Remark 3.2. The formula (3.8) means that the counting function $\omega(G;p,q)$ depends only on the orders of abelian groups A, B, not on their group structures.

4. Weighted integral complementary polynomial

Let A, B be abelian groups. A tension-flow $(f,g)\in\Omega(G,\varepsilon;A,B)$ is said to be *complementary* if $\operatorname{supp} g=\ker f$, also called *parabolic* because (f,g) are not allowed to have the zero value $(0,0)$ and to have nonzero values simultaneously for both f and g. The *complementary space* of (G,ε) is

$$K(G,\varepsilon;A,B):=\{(f,g)\in\Omega(G,\varepsilon;A,B)\mid \operatorname{supp} g=\ker f\}. \tag{4.1}$$

If A, B are a commutative ring R without zero divisors, then

$$K(G,\varepsilon;R)=\{(f,g)\in\Omega\mid f(e)g(e)=0, f(e)+g(e)\neq 0, e\in E\}.$$

If $R=\mathbb{R}$, we simply write $K(G,\varepsilon)$ for $K(G,\varepsilon;\mathbb{R})$. Analogously, we write $T(G,\varepsilon)$, $T_{\mathbb{Z}}(G,\varepsilon)$, $F(G,\varepsilon)$, $F_{\mathbb{Z}}(G,\varepsilon)$, $\Omega(G,\varepsilon)$, $\Omega_{\mathbb{Z}}(G,\varepsilon)$ for $T(G,\varepsilon;\mathbb{R})$, $T(G,\varepsilon;\mathbb{Z})$, $F(G,\varepsilon;\mathbb{R})$, $F(G,\varepsilon;\mathbb{Z})$, $\Omega(G,\varepsilon;\mathbb{Z})$, $\Omega(G,\varepsilon;\mathbb{R})$, respectively.

It is well-known that $T(G,\varepsilon)$ and $F(G,\varepsilon)$ are orthogonal complements in $\mathbb{R}^E$ under the obvious inner product $\langle f,g\rangle=\sum_{e\in E}f(e)g(e)$. So there is a natural vector space isomorphism $\Omega(G,\varepsilon)\simeq\mathbb{R}^E$, $(f,g)\mapsto f+g$. The corresponding lattice satisfies the relation

$$\Omega_{\mathbb{Z}}=T_{\mathbb{Z}}\times F_{\mathbb{Z}}\simeq T_{\mathbb{Z}}\oplus F_{\mathbb{Z}}\subseteq\mathbb{Z}^E. \tag{4.2}$$

Then $\mathbb{Z}^E/(T_{\mathbb{Z}}\oplus F_{\mathbb{Z}})$ is a finite abelian group, whose cardinality is the number of maximal forests of G; see [**4**] (Theorem 9, p.53) and [**5**] (Corollary 12.4, p.219).

We introduce a relatively open 0-1 non-convex polyhedron, called the *complementary polyhedron*,

$$\Delta_{\text{CTF}}(G,\varepsilon):=\{(f,g)\in K(G,\varepsilon): 0<|f+g|<1\}, \tag{4.3}$$

and a relatively open 0-1 convex polytope, called the *complementary polytope* (with respect to ε),

$$\Delta^+_{\text{CTF}}(G,\varepsilon):=\{(f,g)\in K(G,\varepsilon)\mid 0<f+g<1\}. \tag{4.4}$$

Let ϱ be an orientation on G. It is clear that the polyhedron $\Delta_{\text{CTF}}(G,\varepsilon)$ contains the relatively open 0-1 convex polytope

$$\Delta^{\varrho}_{\text{CTF}}(G,\varepsilon):=\{(f,g)\in\Delta_{\text{CTF}}(G,\varepsilon)\ :\ [\varrho,\varepsilon](f+g)>0\}, \tag{4.5}$$

which is lattice polyhedral isomorphic to $\Delta^+_{\text{CTF}}(G,\varrho)$ by the involution map

$$P_{\varrho,\varepsilon}:(\mathbb{R}\times\mathbb{R})^E\to(\mathbb{R}\times\mathbb{R})^E,\quad (f,g)\mapsto(P_{\varrho,\varepsilon}f,P_{\varrho,\varepsilon}g).$$

It is clear that the union $C(G,\varrho)$ of all directed circuits of (G,ϱ) forms a strong subgraph (equivalently totally cyclic). The union $B(G,\varrho)$ of all directed bonds of (G,ϱ) forms an acyclic directed subgraph. Since each directed circuit is edge-disjoint from any directed bond, we see that $B(G,\varrho)$ and $C(G,\varrho)$ are edge-disjoint. Let B_ϱ and C_ϱ be the edges sets of $B(G,\varrho)$ and $C(G,\varrho)$, respectively. Then B_ϱ and C_ϱ are complements in E; see Proposition 3.1 of [**13**]. Thus the digraph (G,ϱ) is naturally decomposed into the edge-disjoint directed subgraphs $B(G,\varrho)$ and $C(G,\varrho)$. We introduce the relatively open 0-1 convex polytopes

$$\Delta^+_{\text{TN}}(G,B_\varrho):=\{f\in T(G,\varrho): 0<f|_{B_\varrho}<1, f|_{C_\varrho}=0\}, \tag{4.6}$$

$$\Delta^+_{\text{FL}}(G,C_\varrho):=\{g\in F(G,\varrho): g|_{B_\varrho}=0, 0<g|_{C_\varrho}<1\}. \tag{4.7}$$

Then the polytope $\Delta^+_{\text{CTF}}(G,\varrho)$ is decomposed into the product

$$\Delta^+_{\text{CTF}}(G,\varrho)=\Delta^+_{\text{TN}}(G,B_\varrho)\times\Delta^+_{\text{FL}}(G,C_\varrho). \tag{4.8}$$

Proposition 4.1. (a) *$B(G,\varrho)$ is acyclic, $C(G,\varrho)$ is totally cyclic.*
(b) *$B_\varrho\cap C_\varrho=\emptyset$, $B_\varrho\cup C_\varrho=E$.*
(c) *$q\Delta^+_{\text{TN}}(G,B_\varrho)\simeq q\Delta^+_{\text{TN}}(G/C_\varrho,\varrho)$, $q\Delta^+_{\text{FL}}(G,C_\varrho)\simeq q\Delta^+_{\text{FL}}(G\backslash B_\varrho,\varrho)$.*

PROOF. (a) It is trivial that $C(G,\varrho)$ is totally cyclic. Suppose $B(G,\varrho)$ contains a directed circuit C. Fix an edge $e_1\in C$ and a directed bond $B_1=[V_1,V_1^c]$ such that $e_1\in B_1$. Since C is a closed path, there exists an edge $e_2\in C$ other than e_1 such that $e_2\in B_1$. Then the orientation of e_2 in C is opposite to the orientation of e_2 in B_1; this is a contradiction. So $B(G,\varrho)$ is acyclic.

(b) It is trivial that $B_\varrho\cap C_\varrho=\emptyset$. Let e be an edge of G such that $e\notin C_\varrho$. Then e cannot be a loop. We may assume that the orientation of e in G is from its one end-vertex u to the other end-vertex v. Let V_1 be the set of vertices from which there is a directed path to the vertex u; the length of the path is allowed to be zero, so that $u\in V_1$. It is clear that $v\notin V_1$; otherwise, there is a directed path P from v to u, then ueP is a directed circuit containing the edge e; this is a contradiction. Thus $[V_1,V_1^c]$ is a cut and contains the edge e. We claim that $[V_1,V_1^c]$ is directed cut from V_1 to V_1^c. Suppose there is an edge $e_1\in[V_1,V_1^c]$ whose orientation is from a vertex $v_1\in V_1^c$ to a vertex $u_1\in V_1$. Let P_1 be a directed path from u_1 to u. Then $v_1e_1P_1$ is a directed path from v_1 to u, so $v_1\in V_1$; this is a contradiction.

(c) Identify the edge set of G/C_ϱ as the edge subset B_ϱ. The first polyhedral isomorphism is given by $f\mapsto f|_{B_\varrho}$, sending lattice points to lattice points. The second polyhedral isomorphism is given by $f\mapsto f|_{C_\varrho}$, also sending lattice points to lattice points. $\square$

Let p,q be positive integers. Recall the polynomial counting functions (introduced in [**13**])

$$\kappa_{\mathbb{Z}}(G;p,q):=|(p,q)\Delta_{\text{CTF}}(G,\varepsilon)\cap(\mathbb{Z}^2)^E|, \tag{4.9}$$
$$\kappa_\varrho(G;p,q):=|(p,q)\Delta^+_{\text{CTF}}(G,\varrho)\cap(\mathbb{Z}^2)^E|, \tag{4.10}$$
$$\bar\kappa_\varrho(G;p,q):=|(p,q)\bar\Delta^+_{\text{CTF}}(G,\varrho)\cap(\mathbb{Z}^2)^E|, \tag{4.11}$$

and the product formulas

$$\kappa_\varrho(G;x,y)=\tau_\varrho(G,B_\varrho;x)\,\varphi_\varrho(G,C_\varrho;y), \tag{4.12}$$
$$\bar\kappa_\varrho(G;x,y)=\bar\tau_\varrho(G,B_\varrho;x)\,\bar\varphi_\varrho(G,C_\varrho;y), \tag{4.13}$$

where $\tau_\varrho(G,B_\varrho;x)$, $\bar\tau_\varrho(G,B_\varrho;x)$, $\varphi_\varrho(G,\varrho;y)$, $\bar\varphi_\varrho(G,\varrho;y)$ are the Ehrhart polynomials of $\Delta^+_{\text{TN}}(G,B_\varrho)$, $\bar\Delta^+_{\text{TN}}(G,B_\varrho)$, $\Delta^+_{\text{FL}}(G,C_\varrho)$, $\bar\Delta^+_{\text{FL}}(G,C_\varrho)$, respectively. For elementary properties about Ehrhart polynomials, we refer to [**10, 11, 27**].

Now for arbitrary integers r,s, we introduce the weighted counting function

$$\psi_{\mathbb{Z}}(G;p,q,r,s):=\sum_{(f,g)\in(\mathbb{Z}^2)^E(p,q)\Delta_{\text{CTF}}(G,\varepsilon)} r^{|\operatorname{supp} f|}s^{|\operatorname{supp} g|}, \tag{4.14}$$

which turns out to be a polynomial function of p,q,r,s, and is independent of the chosen orientation ε, called the *weighted integral complementary polynomial* of G. When $r,s=1$, the polynomial $\psi_{\mathbb{Z}}(G;x,y,1,1)$ reduces to the integral complementary polynomial $\kappa_{\mathbb{Z}}(G;x,y)$ in [**13**] and $I_G(x,y)$ in [**21**]. To understand the information encoded in $\psi_{\mathbb{Z}}(G;x,y,z,w)$, especially the combinatorial interpretation for the values of the polynomial at negative integers of x,y, we further introduce

the weighted counting function

$$\bar{\psi}_{\mathbb{Z}}(G;p,q,r,s) := \sum_{\varepsilon\in\mathcal{O}(G)} r^{|B_\varepsilon|}s^{|C_\varepsilon|}|(\mathbb{Z}^2)^E \cap (p,q)\bar{\Delta}^{+}_{\text{CTF}}(G,\varepsilon)|, \tag{4.15}$$

which turns out to be a polynomial function of non-negative integers p,q and arbitrary integers r,s, called the *weighted dual integral complementary polynomial* of G. When $r,s=1$, the polynomial $\bar{\psi}_{\mathbb{Z}}(G;x,y,1,1)$ reduces to the dual integral complementary polynomial $\bar{\kappa}_{\mathbb{Z}}(G;x,y)$ in [**13**].

Theorem 4.2. *The integer-valued function $\psi_{\mathbb{Z}}(G;p,q,r,s)$ ($\bar{\psi}_{\mathbb{Z}}(G;p,q,r,s)$) is a polynomial function of positive (non-negative) integers p,q and arbitrary integers r,s. Furthermore,*

$$\psi_{\mathbb{Z}}(G;x,y,z,w) = \sum_{\varrho\in\mathcal{O}(G)} z^{|B_\varrho|}\, w^{|C_\varrho|}\kappa_\varrho(G;x,y), \tag{4.16}$$

$$\bar{\psi}_{\mathbb{Z}}(G;x,y,z,w) = \sum_{\varrho\in\mathcal{O}(G)} z^{|B_\varrho|}\, w^{|C_\varrho|}\bar{\kappa}_\varrho(G,x,y), \tag{4.17}$$

and satisfy the Reciprocity Law:

$$\psi_{\mathbb{Z}}(G;-x,-y,z,w) = (-1)^{n(G)}\bar{\psi}_{\mathbb{Z}}(G;x,y,-z,w), \tag{4.18}$$

$$= (-1)^{r(G)}\bar{\psi}_{\mathbb{Z}}(G;x,y,z,-w). \tag{4.19}$$

In particular, $\psi_{\mathbb{Z}}(G;x,y,0,0) = \bar{\psi}_{\mathbb{Z}}(G;x,y,0,0) = 0$,

$$\psi_{\mathbb{Z}}(G;x,y,1,0) = \tau_{\mathbb{Z}}(G,x), \qquad \bar{\psi}_{\mathbb{Z}}(G;x,y,1,0) = \bar{\tau}_{\mathbb{Z}}(G,x), \tag{4.20}$$

$$\psi_{\mathbb{Z}}(G;x,y,0,1) = \varphi_{\mathbb{Z}}(G,y), \qquad \bar{\psi}_{\mathbb{Z}}(G;x,y,0,1) = \bar{\varphi}_{\mathbb{Z}}(G,y), \tag{4.21}$$

$$\psi_{\mathbb{Z}}(G;x,y,1,1) = \kappa_{\mathbb{Z}}(G;x,y), \qquad \bar{\psi}_{\mathbb{Z}}(G;x,y,1,1) = \bar{\kappa}_{\mathbb{Z}}(G;x,y). \tag{4.22}$$

Proof. Let p,q be positive integers. Then (4.16) follows from the decomposition (see Lemma 3.2(a) of [**13**])

$$(\mathbb{Z}^2)^E \cap (p,q)\Delta_{\text{CTF}}(G,\varepsilon) = \bigsqcup_{\varrho\in\mathcal{O}(G)} (\mathbb{Z}^2)^E \cap (p,q)\Delta^{\varrho}_{\text{CTF}}(G,\varepsilon),$$

the identification $\Delta^{\varrho}_{\text{CTF}}(G,\varepsilon) = P_{\varrho,\varepsilon}\Delta^{+}_{\text{CTF}}(G,\varrho)$ (see Lemma 3.2(b) of [**13**]), and the product formula (4.12). The formula (4.17) follows from the definition of $\bar{\psi}_{\mathbb{Z}}$ and (4.13). The Reciprocity Law follows from the reciprocity law of the polynomials ψ_ϱ and $\bar{\psi}_\varrho$ in the following lemma. □

Lemma 4.3. *For each orientation ϱ on G, define*

$$\psi_\varrho(G;x,y,z,w) : = z^{|B_\varrho|}w^{|C_\varrho|}\kappa_\varrho(G;x,y), \tag{4.23}$$

$$\bar{\psi}_\varrho(G;x,y,z,w) : = z^{|B_\varrho|}w^{|C_\varrho|}\bar{\kappa}_\varrho(G;x,y). \tag{4.24}$$

Then ψ_ϱ and $\bar{\psi}_\varrho$ satisfy the Reciprocity Law:

$$\psi_\varrho(G;-x,-y,z,w) = (-1)^{n(G)}\bar{\psi}_\varrho(G;x,y,-z,w) \tag{4.25}$$

$$= (-1)^{r(G)}\bar{\psi}_\varrho(G;x,y,z,-w). \tag{4.26}$$

PROOF. Recall $\kappa_\varrho(G;-x,-y)=(-1)^{r(G)+|C_\varrho|}\bar{\kappa}_\varrho(G,x,y)$ from Proposition 3.3(c) of [**13**]. It follows that

$$\begin{aligned}\psi_\varrho(G;-x,-y,z,w) &= (-1)^{r(G)+|C_\varrho|}z^{|B_\varrho|}w^{|C_\varrho|}\bar{\kappa}_\varrho(G;x,y)\\ &= (-1)^{r(G)}z^{|B_\varrho|}(-w)^{|C_\varrho|}\bar{\kappa}_\varrho(G;x,y)\\ &= (-1)^{r(G)}\bar{\psi}_\varrho(G;x,y,z,-w),\end{aligned}$$

which is (4.26). Using $r(G)+n(G)=|E|$ and $|B_\varrho|+|C_\varrho|=|E|$, we obtain (4.25) as well. □

5. Weighted modular complementary polynomial

In this section we pass from the counting of integral tension-flows with weights to the counting of modular tension-flows with weights. Let p,q be positive integers. For brevity, we write $\mathbb{Z}_p$ for $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}_q$ for $\mathbb{Z}/q\mathbb{Z}$. There is a (p,q)-*modular map*

$$\mathrm{Mod}_{p,q}:\mathbb{R}^E\times\mathbb{R}^E\to(\mathbb{R}/p\mathbb{Z})^E\times(\mathbb{R}/q\mathbb{Z})^E, \tag{5.1}$$

defined for $(f,g)\in\mathbb{R}^E\times\mathbb{R}^E$ and $e\in E$ by

$$\mathrm{Mod}_{p,q}(f,g)(e)=(f(e)\bmod p,\ g(e)\bmod q). \tag{5.2}$$

Let ε,ϱ be orientations on G and S an edge subset of G. Let $Q^p_{\varrho,\varepsilon,S}:[0,p]^E\to[0,p]^E$ be an involution relative to S, defined for $f\in[0,p]^E$ and $e\in E$ by

$$(Q^p_{\varrho,\varepsilon,S}f)(e)=\begin{cases} p-f(x) & \text{if } \ e\in S,\ \varrho(e)\neq\varepsilon(e),\\ f(x) & \text{otherwise.}\end{cases} \tag{5.3}$$

Then the involutions $Q^p_{\varrho,\varepsilon,S}$ and $Q^q_{\varrho,\varepsilon,S^c}$ give arise to an involution

$$Q^{p,q}_{\varrho,\varepsilon,S}:[0,p]^E\times[0,q]^E\to[0,p]^E\times[0,q]^E, \tag{5.4}$$

given by $Q^{p,q}_{\varrho,\varepsilon,S}(f,g)=(Q^p_{\varrho,\varepsilon,S}f,Q^q_{\varrho,\varepsilon,S^c}g)$, where $(f,g)\in[0,p]^E\times[0,q]^E$.

Recall the equivalence relation on the set $\mathcal{O}(G)$ of all orientations on G. Two orientations ε,ϱ on G are said to be *cut-Eulerian equivalent*, written $\varepsilon\sim_{\mathrm{CE}}\varrho$, if the subgraph induced by the edge set

$$E(\varepsilon\neq\varrho):=\{e\in E(G)\mid\varepsilon(e)\neq\varrho(e)\},$$

is a disjoint union of directed circuits and directed bonds, with the orientation either ε or ϱ. Indeed, $\sim_{\mathrm{CE}}$ is an equivalence relation on $\mathcal{O}(G)$; see Lemma 4.2(a) of [**13**]. We denote by $[\mathcal{O}(G)]$ the set of all cut-Eulerian equivalence classes of $\mathcal{O}(G)$.

It is shown that the restriction $\mathrm{Mod}_{p,q}:(p,q)\Delta_{\mathrm{CTF}}(G,\varepsilon)\to K(G,\varepsilon;\mathbb{Z}_p,\mathbb{Z}_q)$ is surjective; see Lemma 4.4 of [**13**]. For each $(f,g)\in(p,q)\Delta^\varrho_{\mathrm{CTF}}(G,\varepsilon)$, the inverse image $(p,q)\Delta_{\mathrm{CTF}}(G,\varepsilon)\cap\mathrm{Mod}^{-1}_{p,q}\mathrm{Mod}_{p,q}(f,g)$ consists of the elements of the form $P_{\varepsilon,\alpha}Q^{p,q}_{\alpha,\varrho,S}P_{\varrho,\varepsilon}(f,g)$, where $\alpha\in[\varrho]$; see Lemma 4.5 of [**13**]. Moreover, for each orientation ϱ on G, $\#[\varrho]$ equals the number of 0-1 complementary tension-flows of (G,ϱ); see Lemma 4.6 of [**13**]. Thus, writing in formulas, we have

$$\begin{aligned}\#[\varrho]=\bar{\kappa}_\varrho(G;1,1) &= |(\mathbb{Z}^2)^E\cap\bar{\Delta}^+_{\mathrm{CTF}}(G,\varrho)| &(5.5)\\ &= |(p,q)\Delta_{\mathrm{CTF}}(G,\varepsilon)\cap\mathrm{Mod}^{-1}_{p,q}\mathrm{Mod}_{p,q}(f,g)|, &(5.6)\end{aligned}$$

where $(f,g)\in(p,q)\Delta^\varrho_{\mathrm{CTF}}(G,\varepsilon)$.

Let A, B be finite abelian groups of orders $|A| = p, |B| = q$. For arbitary integers r, s, we introduce a weighted counting function

$$\psi(G; p, q, r, s) := \sum_{(f,g)\in K_{\rm nz}(G,\varepsilon;A,B)} r^{|\operatorname{supp} f|} s^{|\operatorname{supp} g|}, \tag{5.7}$$

which turns out to be a polynomial function of positive integers p, q and integers r, s, called the *weighted complementary polynomial* of G. To interpret the values of the polynomial $\psi(G; x, y, z, t)$ when x, y are negative integers, we introduce the following polynomial counting function

$$\bar\psi(G; p, q, r, s) := \sum_{[\varrho]\in[\mathcal{O}(G)]} r^{|B_\varrho|} s^{|C_\varrho|} |(p, q)\bar\Delta^{+}_{\rm CTF}(G, \varrho) \cap (\mathbb{Z}^2)^E| \tag{5.8}$$

of non-negative integers p, q and arbitrary integers r, s, called the *dual weighted complementary polynomial* of G. When $r, s = 1$, $\psi(G; p, q, r, s)$ reduces to $\kappa(G; p, q)$ in [**13**] and $F_G(p, q)$ in [**21**], and $\bar\psi(G; p, q, r, s)$ reduces to $\bar\kappa(G; p, q)$ in [**13**].

Theorem 5.1. *The counting function $\psi(G; p, q, r, s)$ $(\bar\psi(G; p, q, r, s))$ is a polynomial function of positive (non-negative) integers p, q and integers r, s, and can be written as*

$$\psi(G; x, y, z, w) = \sum_{\varrho\in[\mathcal{O}(G)]} z^{|B_\varrho|} w^{|C_\varrho|} \kappa_\varrho(G; x, y), \tag{5.9}$$

$$\bar\psi(G; x, y, z, w) = \sum_{\varrho\in[\mathcal{O}(G)]} z^{|B_\varrho|} w^{|C_\varrho|} \bar\kappa_\varrho(G; x, y). \tag{5.10}$$

Moreover, ψ and $\bar\psi$ satisfy the Reciprocity Law:

$$\psi(G; -x, -y, z, w) = (-1)^{n(G)} \bar\psi(G; x, y, -z, w), \tag{5.11}$$

$$= (-1)^{r(G)} \bar\psi(G; x, y, z, -w). \tag{5.12}$$

In particular, $\psi(G; x, y, 0, 0) = \bar\psi(G; x, y, 0, 0) = 0$, and

$$\psi(G; x, y, 1, 0) = \tau(G, x), \qquad \bar\psi(G; x, y, 1, 0) = \bar\tau(G, x), \tag{5.13}$$

$$\psi(G; x, y, 0, 1) = \varphi(G, y), \qquad \bar\psi(G; x, y, 0, 1) = \bar\varphi(G, y), \tag{5.14}$$

$$\psi(G; x, y, 1, 1) = \kappa(G; x, y), \qquad \bar\psi(G; x, y, 1, 1) = \bar\kappa(G; x, y). \tag{5.15}$$

Proof. Fix an orientation ϱ on G; let $\Delta_{[\varrho]} := \bigsqcup_{\rho\in[\varrho]} (p, q)\Delta^{\rho}_{\rm CTF}(G, \varepsilon)$. By Lemma 4.7 of [**13**], we have

$$(p, q)\Delta_{[\varrho]} = (p, q)\Delta_{\rm CTF}(G, \varepsilon) \cap {\rm Mod}^{-1}_{p,q}{\rm Mod}_{p,q}(p, q)\Delta^{\varrho}_{\rm CTF}(G, \varepsilon).$$

Since the orientation ϱ can be replaced by any orientation ρ that is cut-Eulerian equivalent to ϱ, we further have

$$(p, q)\Delta_{[\varrho]} = (p, q)\Delta_{\rm CTF}(G, \varepsilon) \cap {\rm Mod}^{-1}_{p,q}{\rm Mod}_{p,q}(p, q)\Delta_{[\varrho]}. \tag{5.16}$$

On the one hand, counting the lattice points along the inverse fiber of ${\rm Mod}_{p,q}$, we obtain the product

$$|(\mathbb{Z}^2)^E \cap (p, q)\Delta_{[\varrho]}| = |{\rm Mod}_{p,q}(\mathbb{Z}^2)^E \cap (p, q)\Delta_{[\varrho]}| \cdot \bar\kappa_\varrho(G; 1, 1).$$

On the other hand, recall that $\kappa_\rho(G; p, q) = \kappa_\varrho(G; p, q)$ if $\rho \sim_{\rm CE} \varrho$; then by definition of $\Delta_{[\varrho]}$, we obtain another product

$$|(\mathbb{Z}^2)^E \cap (p, q)\Delta_{[\varrho]}| = \kappa_\varrho(G; p, q) \cdot \bar\kappa_\varrho(G; 1, 1).$$

It then follows that

$$\kappa_\varrho(G;p,q) = |\mathrm{Mod}_{p,q}(\mathbb{Z}^2)^E \cap (p,q)\Delta_{[\varrho]}|.$$

Now recall the decomposition

$$K(G,\varepsilon;\mathbb{Z}_p,\mathbb{Z}_q) = \bigsqcup_{[\varrho]\in[\mathcal{O}(G)]} \mathrm{Mod}_{p,q}(p,q)\Delta_{[\varrho]} \cap (\mathbb{Z}^2)^E.$$

For each $(f,g)\in(p,q)\Delta^{\rho}_{\text{CTF}}(G,\varepsilon)$, let $(\tilde f,\tilde g) = \mathrm{Mod}_{p,q}(f,g)$. Then

$$\operatorname{supp}\tilde f = \operatorname{supp} f = B_\varrho, \quad \operatorname{supp}\tilde g = \operatorname{supp} g = C_\varrho.$$

Thus

$$\psi(G;p,q,r,s) = \sum_{\varrho\in[\mathcal{O}(G)]} r^{|B_\varrho|} s^{|C_\varrho|}\kappa_\varrho(G;p,q),$$

The reciprocity law follows from the reciprocity law of ψ_ϱ and $\bar\psi_\varrho$; see Lemma 4.3. □

We end up this section by stating an interpretation for all values of the Tutte polynomial. Part (i) is already given in [**13**].

Corollary 5.2. *Let* $\mathrm{SDR}[\mathcal{O}(G)]$ *be a set of distinct representatives of cut-Eulerian equivalence classes of* $[\mathcal{O}(G)]$. *Let* p,q *be positive integers. Then*

(i) $T(G;p,q)$ *counts the number of triples* (ϱ,f,g), *where* $\varrho\in\mathrm{SDR}[\mathcal{O}(G)]$, (f,g) *is an integer-valued tension-flow of* (G,ϱ) *such that*

$$0\le f<p,\ 0\le g<q.$$

(ii) $T(G;-p,q)$ *counts the number of signed triples* $(-1)^{r(G/C_\varrho)}(\varrho,f,g)$, *where* $\varrho\in\mathrm{SDR}[\mathcal{O}(G)]$, (f,g) *is an integer-valued tension-flow of* (G,ϱ) *such that*

$$0<f|_{B_\varrho}\le p,\ f|_{C_\varrho}=0,\ 0\le g<q.$$

(iii) $T(G;p,-q)$ *counts the number of signed triples* $(-1)^{n\langle C_\varrho\rangle}(\varrho,f,g)$, *where* $\varrho\in\mathrm{SDR}[\mathcal{O}(G)]$, (f,g) *is an integer-valued tension-flow of* (G,ϱ) *such that*

$$0\le f<p,\ g|_{B_\varrho}=0,\ 0<g|_{C_\varrho}\le q.$$

(iv) $T(G;-p,-q)$ *counts the number of signed triples* $(-1)^{r(G)+|C_\varrho|}(\varrho,f,g)$, *where* $\varrho\in\mathrm{SDR}[\mathcal{O}(G)]$, (f,g) *is an integer-valued tension-flow of* (G,ϱ) *such that*

$$0<f|_{B_\varrho}\le p,\ f|_{C_\varrho}=0,\ g|_{B_\varrho}=0,\ 0<g|_{C_\varrho}\le q.$$

Proof. For the proof of part (i), see Theorem 1.3 and Corollary 1.4 in [**13**]. Since $T(G;x,y)=\bar\kappa(G;x-1,y-1)$ and the decomposition formula (5.10), then for the Tutte polynomial T we have the expansion

$$T(G;x,y) = \sum_{\varrho\in\mathrm{SDR}[\mathcal{O}(G)]} \bar\tau_\varrho(G/C_\varrho,x-1)\,\bar\varphi_\varrho(\langle C_\varrho\rangle,y-1).$$

Recall the the reciprocity laws for τ_ϱ and φ_ϱ; we have

$$\tau_\varrho(G;-p,q) = \sum_{\varrho\in\mathrm{SDR}[\mathcal{O}(G)]} (-1)^{r(G/C_\varrho)}\tau_\varrho(G,p+1)\,\bar\varphi_\varrho(G\backslash B_\varrho,q-1),$$

$$\tau_\varrho(G;p,-q) = \sum_{\varrho\in\mathrm{SDR}[\mathcal{O}(G)]} (-1)^{n\langle C_\varrho\rangle}\bar\tau_\varrho(G,p)\,\varphi_\varrho(G\backslash B_\varrho,-q-1),$$

$$\tau_\varrho(G;-p,-q)=\sum_{\varrho\in \mathrm{SDR}[\mathcal{O}(G)]}(-1)^{r(G/C_\varrho)+n\langle C_\varrho\rangle}\tau_\varrho(G,p+1)\,\varphi_\varrho(G\backslash B_\varrho,q+1).$$

Since $r(G/C_\varrho)=r(G)-r\langle C_\varrho\rangle$ and $r\langle C_\varrho\rangle+n\langle C_\varrho\rangle=|C_\varrho|$, it follows that $r(G/C_\varrho)+n\langle C_\varrho\rangle=r(G)+|C_\varrho|$. By definition of $\tau_\varrho,\bar\tau_\varrho,\varphi_\varrho,\bar\varphi_\varrho$ (see (4.12) and (4.13)), parts (ii), (iii), and (iv) are equivalent to the above three expansion formulas. □

Remark Notice that $T(G;-p,-q)=R(G;-(p+1),-(q+1))$ for positive integers p,q. The expansion formula (6.3) gives a different interpretation for $T(G;-p,-q)$, counting the number of signed tension-flows (f,g) of (G,ε) with values in $\mathbb{Z}_{p+1}$ and $\mathbb{Z}_{q+1}\times\mathbb{Z}_{q+1}$ and with the sign $(-1)^{|\mathrm{supp}\,g|}$.

6. Other weighted Tutte type polynomials

Recall Whitney's *rank generating polynomial* for a graph $G=(V,E)$ is defined by

$$R(G;x,y)=\sum_{X\subseteq E}x^{r(G)-r\langle X\rangle}y^{n\langle X\rangle}, \tag{6.1}$$

where $\langle X\rangle=(V,X)$ is the spanning subgraph with the edge set X. The *Tutte polynomial* of G is defined by the substitution of variables:

$$T(G;x,y)=R(G;x-1,y-1).$$

Let E_n be the graph with n vertices and empty edge set. The Tutte polynomial can be computed by the recursion:

$$T\big(E_n;x,y\big)=1,$$

$$T(G;x,y)=\begin{cases} yT(G\backslash e;x,y) & \text{if } e \text{ is a loop,}\\ xT(G/e;x,y) & \text{if } e \text{ is a cut-edge,}\\ T(G\backslash e;x,y)+T(G/e;x,y) & \text{otherwise.}\end{cases}$$

Let A,B be finite abelian groups of orders $|A|=p$, $|B|=q$, and $\Omega:=\Omega(G,\varepsilon;A,B)$. Then $R(G;x,y)$ has the following interpretations:

$$R(G;p,q)=\sum_{\substack{(f,g)\in\Omega\\ \mathrm{supp}\,g\subseteq\ker f}}2^{|\ker f-\mathrm{supp}\,g|}, \tag{6.2}$$

$$R(G;-p,-q)=(-1)^{r(G)}\sum_{\substack{(f,g)\in\Omega\\ \mathrm{supp}\,g=\ker f}}(-1)^{|\mathrm{supp}\,g|}. \tag{6.3}$$

Both (6.2) and (6.3) are due to Reiner [**24**], obtained from a convolution formula for the Tutte polynomial [**22**]. A simple and direct proof for (6.2) and (6.3) is provided by the present author; see Proposition 5.1 of [**13**].

Searching on-line recently (when revising the paper), we found a combinatorial interpretation for the Tutte polynomial by Breuer and Sanyal [**6**] at positive integers p,q: $T(G;p,q)$ *is the number of triplets* (f,g,ϱ)*, where* (f,g) *is a* $\mathbb{Z}_p\times\mathbb{Z}_q$*-tension-flow of* (G,ε) *such that* $\mathrm{supp}\,g\subseteq\ker f$*, and* ϱ *is a geometric reorientation on the edge subset* $\ker f-\mathrm{supp}\,g$. Here "geometric" means that each loop is allowed to have exactly two orientations. Note that we assume that each loop has only one orientation "combinatorially." This interpretation is naively a reformulation of (6.2), for each edge of $\ker f-\mathrm{supp}\,g$ has exactly two choices to be reoriented.

Let $X,Y\subseteq E$ be edge subsets. Recall $\Omega_{X,Y}=T_X\times F_Y$. We define the subset

$$\Omega^0_{X,Y}:=\{(f,g)\in\Omega(G,\varepsilon;A,B)\mid \ker f=X,\ker g=Y\}.$$

It is clear that $\Omega_{X,Y}$ is the disjoint union $\bigsqcup_{X\subseteq Z,\,Y\subseteq W}\Omega^0_{Z,W}$ so that

$$1_{\Omega_{X,Y}}=\sum_{X\subseteq Z,\,Y\subseteq W}1_{\Omega^0_{Z,W}}.$$

By the Möbius inversion, we have

$$1_{\Omega^0_{X,Y}}=\sum_{X\subseteq Z,\,Y\subseteq W}(-1)^{|Z-X|+|W-Y|}\,1_{\Omega_{Z,W}}.$$

In particular, if $|A|=p$, $|B|=q$, we then further have

$$|\Omega^0_{X,Y}|=\sum_{X\subseteq Z,\,Y\subseteq W}(-1)^{|Z-X|+|W-Y|}p^{r\langle E\rangle-r\langle Z\rangle}q^{r\langle W^c\rangle}.$$

Theorem 6.1. *Let ν be a valuation on the Boolean algebra of the tension-flow space $\Omega(G,\varepsilon;A,B)$. Then*

$$\iint_{\operatorname{supp} f\subseteq\ker g}u^{|\ker f|}v^{|\operatorname{supp} g|}\,\mathrm{d}\nu(f,g)=\sum_{Y\subseteq X\subseteq E}(uv)^{|Y|}\,(u-uv-1)^{|X-Y|}\,\nu(\Omega_{X,Y^c}).\tag{6.4}$$

Proof. The left-hand side of (6.4) can be written as

$$\begin{aligned}\text{LHS}\;&=\;\sum_{Y\subseteq X\subseteq E}u^{|X|}v^{|Y|}\,\nu(\Omega^0_{X,Y^c})\\&=\;\sum_{Y\subseteq X\subseteq E}u^{|X|}v^{|Y|}\sum_{X\subseteq Z,\,W\subseteq Y}(-1)^{|Z-X|+|Y-W|}\,\nu(\Omega_{Z,W^c})\\&=\;\sum_{W\subseteq Z\subseteq E}(-1)^{|Z|+|W|}\,\nu(\Omega_{Z,W^c})\sum_{W\subseteq Y\subseteq X\subseteq Z}(-u)^{|X|}(-v)^{|Y|}.\end{aligned}$$

For the fixed edge subsets $Z,W\subseteq E$, applying the binomial theorem, we see that

$$\begin{aligned}\sum_{W\subseteq Y\subseteq X\subseteq Z}(-u)^{|X|}(-v)^{|Y|}&=\sum_{W\subseteq X\subseteq Z}(-u)^{|X|}(-v)^{|W|}\sum_{W\subseteq Y\subseteq X}(-v)^{|Y-W|}\\&=\sum_{W\subseteq X\subseteq Z}(-u)^{|X|}(-v)^{|W|}(1-v)^{|X-W|}\\&=(uv)^{|W|}\sum_{W\subseteq X\subseteq Z}(uv-u)^{|X-W|}\\&=(uv)^{|W|}(uv-u+1)^{|Z-W|}.\end{aligned}$$

Put the identity into the LHS; we obtain (6.4). □

Corollary 6.2. *Let A,B be finitely generated abelian groups or infinite fields. Let λ be the product valuation of the unique valuations*

$$\lambda_1:\mathscr{B}(T(G,\varepsilon;A))\to\mathbb{Q}[x],\quad \lambda_2:\mathscr{B}(F(G,\varepsilon;B))\to\mathbb{Q}[y].$$

Then $\lambda(\Omega_{X,Y^c})=x^{r\langle E\rangle-r\langle X\rangle}y^{n\langle Y\rangle}$. Moreover, if $u=2$ and $v=1/2$, then the left-hand side of (6.4) reduces to the rank generating polynomial

$$R(G;x,y)=\iint_{\operatorname{supp} g\subseteq\ker f}2^{|\ker f-\operatorname{supp} g|}\,\mathrm{d}\lambda(f,g).$$

Theorem 6.3. *Let ν be a valuation on the Boolean algebra of the tension-flow space $\Omega(G, \varepsilon; A, B)$. Then*

$$\iint_{\operatorname{supp} f = \ker g} u^{|\ker f|}\, \mathrm{d}\nu(f,g) = \sum_{Y \subseteq X \subseteq E} u^{|Y|}(-u-1)^{|X-Y|}\, \nu(\Omega_{X,Y^c}). \tag{6.5}$$

In particular, if the abelian groups A, B are finitely generated or infinite fields, ν is the product valuation λ, and $u = -1$, then the rank generating polynomial $R(G; x, y)$ can be expressed as

$$R(G; -x, -y) = (-1)^{r(G)} \iint_{\ker f = \operatorname{supp} g} (-1)^{|\ker f|}\, \mathrm{d}\lambda(f,g). \tag{6.6}$$

Proof. The left-hand side of (6.5) can be written as

$$\begin{aligned} \text{LHS} &= \sum_{X \subseteq E} u^{|X|}\, \nu\big(\Omega^0_{X,X^c}\big) \\ &= \sum_{X \subseteq E} u^{|X|} \sum_{W \subseteq X \subseteq Z} (-1)^{|Z-X|+|X-W|}\, \nu\big(\Omega_{Z,W^c}\big) \\ &= \sum_{W \subseteq Z \subseteq E} (-1)^{|Z-W|}\, \nu\big(\Omega_{Z,W^c}\big) \sum_{W \subseteq X \subseteq Z} u^{|X|}. \end{aligned}$$

Since $\sum_{W \subseteq X \subseteq Z} u^{|X|} = u^{|W|}(u+1)^{|Z-W|}$, the identity (6.5) follows immediately.

Let $\nu = \lambda$ be the unique product valuation and $u = -1$. Note that $|X| = r\langle X\rangle + n\langle X\rangle$; the right-hand side of (6.5) becomes

$$\begin{aligned} \sum_{X \subseteq E} (-1)^{|X|}\, \lambda\big(\Omega_{X,X^c}\big) &= \sum_{X \subseteq E} (-1)^{|X|} x^{r\langle E\rangle - r\langle X\rangle} y^{n\langle X\rangle} \\ &= (-1)^{r\langle E\rangle} \sum_{X \subseteq E} (-x)^{r\langle E\rangle - r\langle X\rangle} (-y)^{n\langle X\rangle} \\ &= (-1)^{r\langle E\rangle} R(G; -x, -y). \end{aligned}$$

□

The following corollary is about the expansion formula for the weighted complementary polynomial $\psi(G; x, y, z, w)$ when the valuation ν is taken to be the unique valuation λ.

Corollary 6.4. *Let ν be a valuation on the Boolean algebra of the tension-flow space $\Omega(G, \varepsilon; A, B)$. Then*

$$\begin{aligned} &\iint_{\operatorname{supp} g = \ker f} z^{|\operatorname{supp} f|}\, w^{|\operatorname{supp} g|}\, \mathrm{d}\nu(f,g) = \\ &\qquad \sum_{Y \subseteq X \subseteq E} z^{|E-X|}\, w^{|Y|}(-z-w)^{|X-Y|}\, \nu(\Omega_{X,Y^c}). \end{aligned} \tag{6.7}$$

Proof. In formula (6.5), let $u = w/z$. Since $\operatorname{supp} g = \ker f$, then $u^{\ker f} = z^{-|E|}\, z^{|\operatorname{supp} f}\, w^{\operatorname{supp} g}$. The left-hand side of (6.5) is

$$\text{LHS} = z^{-|E|} \iint_{\operatorname{supp} g = \ker f} z^{|\operatorname{supp} f|}\, w^{|\operatorname{supp} g|}\, \mathrm{d}\nu(f,g).$$

Note that $u^{|Y|}(-u-1)^{|X-Y|}=z^{-|X|}w^{|Y|}(-z-w)^{|X-Y|}$. The right-hand side of (6.5) is

$$\mathrm{RHS}=\sum_{Y\subseteq X\subseteq E} z^{-|X|}w^{|Y|}(-z-w)^{|X-Y|}\nu(\Omega_{X,Y^c}).$$

We thus obtain the weighted integration formula (6.7). $\square$

Now we derive weighted integral formulas for the *elliptic* case: $\ker f\subseteq \operatorname{supp} g$. It is called elliptic because (f,g) is allowed to have the zero value $(0,0)$ an edge.

Theorem 6.5. *Let ν be a valuation on the Boolean algebra of the tension-flow space $\Omega(G,\varepsilon;A,B)$. Then*

$$\iint_{\ker f\subseteq \operatorname{supp} g} u^{|\ker f|}v^{|\operatorname{supp} g|}\mathrm{d}\nu(f,g)=\sum_{Z,W\subseteq E}(-1)^{|Z|}h(u,v)\,\nu(\Omega_{Z,W^c}),\tag{6.8}$$

where $h(u,v)=v^{|W|}(1-u)^{|Z\cap W|}(1-v)^{|E-Z\cup W|}(uv-v+1)^{|Z-W|}$. In particular, if $u=1,v=1$, then (6.8) reduces to

$$\iint_{\ker f\subseteq \operatorname{supp} g}\mathrm{d}\nu(f,g)=\sum_{Z\subseteq E}(-1)^{|Z|}\nu(\Omega_Z).\tag{6.9}$$

Proof. The left-hand side of (6.8) can be written as

$$\begin{aligned}
\mathrm{LHS}&=\sum_{X\subseteq Y\subseteq E} u^{|X|}v^{|Y|}\nu(\Omega_{X,Y^c})\\
&=\sum_{X\subseteq Y\subseteq E} u^{|X|}v^{|Y|}\sum_{X\subseteq Z,\,W\subseteq Y}(-1)^{|Z-X|+|Y-W|}\nu(\Omega_{Z,W^c})\\
&=\sum_{Z,W\subseteq E}(-1)^{|Z|+|W|}\nu(\Omega_{Z,W^c})\sum_{X\subseteq Z,\,W\subseteq Y,X\subseteq Y}(-u)^{|X|}(-v)^{|Y|}.
\end{aligned}$$

The sum $I:=\sum_{X\subseteq Z,\,W\subseteq Y,X\subseteq Y}(-u)^{|X|}(-v)^{|Y|}$ can be simplified as

$$\begin{aligned}
I&=\sum_{X\subseteq Z}(-u)^{|X|}\sum_{W\cup X\subseteq Y}(-v)^{|Y|}\\
&=\sum_{X\subseteq Z}(-u)^{|X|}(-v)^{|W\cup X|}(1-v)^{|E-W\cup X|}\\
&=(1-v)^{|E|}\sum_{X\subseteq Z}(-u)^{|X|}\Big(\frac{v}{v-1}\Big)^{|W\cup X|}.
\end{aligned}$$

Since W,Z are fixed edge subsets, the edge subset $X\subseteq Z$ can be decomposed into subsets $X_1:=X\cap W\cap Z$ and $X_2:=X\cap(Z-W)$. Set $a:=-u$ and $b:=uv/(1-v)$. Then $u=-a$, $v=b/(b-a)$. We have

$$\begin{aligned}
\sum_{X\subseteq Z}(-u)^{|X|}\Big(\frac{v}{v-1}\Big)^{|W\cup X|}&=\sum_{\substack{X_1\subseteq W\cap Z\\ X_2\subseteq Z-W}} a^{|X_1|+|X_2|}\Big(\frac{b}{a}\Big)^{|W|+|X_2|}\\
&=\Big(\frac{b}{a}\Big)^{|W|}\sum_{\substack{X_1\subseteq W\cap Z\\ X_2\subseteq Z-W}} a^{|X_1|}b^{|X_2|}\\
&=\Big(\frac{b}{a}\Big)^{|W|}(1+a)^{|W\cap Z|}(1+b)^{|Z-W|}.
\end{aligned}$$

The sum I is then figured out as the following

$$\begin{aligned} I &= (1-v)^{|E|}\Big(\frac{v}{v-1}\Big)^{|W|}(1-u)^{|W\cap Z|}\Big(1+\frac{uv}{1-v}\Big)^{|Z-W|} \\ &= (-v)^{|W|}(1-u)^{|W\cap Z|}(1-v)^{|E-W\cup Z|}(uv-v+1)^{|Z-W|}. \end{aligned}$$

We thus obtain the formula (6.8).

Let $u=1, v=1$. Then $(1-u)^{|Z\cap W|}=0$ unless $Z\cap W=\emptyset$; $(1-v)^{|E-Z\cup W|}=0$ unless $Z\cup W=E$. So $h(u,v)\neq 0$ if and only if $Z=W^c$. In fact, $h(u,v)=1$ when $Z=W^c$. The formula (6.9) follows immediately. ☐

Corollary 6.6. *Let ν be a valuation on the Boolean algebra of the tension-flow space $\Omega(G,\varepsilon;A,B)$. Then*

$$\iint_{\operatorname{supp} g\subseteq \ker f} u^{|\operatorname{supp} g|}(u+1)^{|\ker f-\operatorname{supp} g|}\,\mathrm{d}\nu = \sum_{X\subseteq E} u^{|X|}\nu(\Omega_{X,X^c}).$$

Proof. Rearranged the variables, the left-hand side is given by

$$\begin{aligned} \text{LHS} &= \iint_{\operatorname{supp} g\subseteq \ker f} (u+1)^{|\ker f|}\left(\frac{u}{u+1}\right)^{|\operatorname{supp} g|}\mathrm{d}\nu \\ &= \sum_{Y\subseteq X\subseteq E} u^{|Y|}0^{|X-Y|}\nu(\Omega_{X,Y^c}) \\ &= \sum_{X\subseteq E} u^{|X|}\nu(\Omega_{X,X^c}). \end{aligned}$$

☐

Appendix 1: Relative Boolean algebra

Relative Boolean algebras are slightly different from Boolean algebras; the former is closed under the set operation of relatively complement, the latter is closed under complement. The following two lemmas provide with us a general pattern for an element in a relative Boolean algebra and the pattern for elements in the product of relative Boolean algebras.

Lemma 6.7. *Let $\mathscr{B}$ be a relative Boolean algebra generated by an intersectional class $\mathcal{L}$. Then $\mathscr{B}$ consists of sets of the form*

$$\bigcup_{i\in I}\Big(A_i-\bigcup_{k\in I_i}A_{i,k}\Big), \tag{6.10}$$

where $A_i, A_{i,k}\in\mathcal{L}$, and the union extended over I can be made into disjoint.

Proof. Let $\mathcal{B}'$ denote the class of sets of the form (6.10). It is clear that $\mathcal{B}'\subseteq\mathcal{B}$. It suffices to show that $\mathcal{B}'$ is a relative Boolean algebra. Let

$$A=\bigcup_{i\in I}\Big(A_i-\bigcup_{k\in I_i}A_{i,k}\Big),\quad B=\bigcup_{j\in J}\Big(B_j-\bigcup_{l\in J_j}B_{j,l}\Big),$$

where $A_i, A_{i,k}, B_j, B_{j,l}\in\mathcal{L}$, the index sets I, J are finite, and the subindex sets I_i, J_j are finite for all $i\in I, j\in J$. Clearly, the union $A\cup B$ is of the form (6.10).

Since $A_i \cap B_j \in \mathcal{L}$, the intersection

$$\begin{aligned} A \cap B &= \bigcup_{i\in I, j\in J} \Big[\Big(A_i - \bigcup_{k\in I_i} A_{i,k}\Big) \cap \Big(B_j - \bigcup_{k\in J_j} B_{j,l}\Big)\Big] \\ &= \bigcup_{i\in I, j\in J} \Big(A_i \cap B_j - \bigcup_{k\in I_i, l\in J_j} A_{i,k} \cup B_{j,l}\Big) \end{aligned}$$

is also of the form (6.10). So $\mathcal{B}'$ is closed under finite intersections and finite unions. As for the relative complement, note that

$$B - A = \bigcup_{j\in J} \Big[B_j \cap \Big(\bigcup_{l\in J_j} B_{j,l}\Big)^c \cap A^c\Big].$$

Since $\mathcal{B}'$ is closed under finite unions, it suffices to show that for each fixed index j in J, the intersection

$$B_j \cap \Big(\bigcup_{l\in J_j} B_{j,l}\Big)^c \cap A^c = \bigcap_{l\in J_j} B_j \cap (B_{j,l} \cup A)^c.$$

belongs to $\mathcal{B}'$. In fact, if J_j is empty, the union $\cup_{l\in\emptyset} B_{j,l}$ is the empty set, and the intersection is $B_j \cap A^c$; if J_j is not empty, then it is enough to show that $B_j \cap (B_{j,l} \cup A)^c$ belongs to $\mathcal{B}'$, for $\mathcal{B}'$ is closed under finite intersections. Since $B_j, B_{j,l} \in \mathcal{L}$ and $B_{j,l} \cup A$ is of the form (6.10), we are left to show only that $B_0 \cap A^c$ is a member of $\mathcal{B}'$ for $B_0 \in \mathcal{L}$ and $A \in \mathcal{B}'$.

Now let A be written in the form (6.10) and B_0 a member of $\mathcal{L}$. Let us write the index set $I = \{1, 2, \ldots, n\}$. Note that A can be written as

$$\begin{aligned} A^c &= \bigcap_{i\in I} \Big(A_i - \bigcup_{k\in I_i} A_{i,k}\Big)^c \\ &= \bigcap_{i\in I} \Big(A_i^c \cup \bigcup_{k\in I_i} A_{i,k}\Big) \\ &= \Big(A_1^c \cup \bigcup_{k\in I_1} A_{1,k}\Big) \cap \cdots \cap \Big(A_n^c \cup \bigcup_{k\in I_n} A_{n,k}\Big). \end{aligned}$$

Expanding this intersection into unions, we have

$$\begin{aligned} B_0 \cap A^c &= \bigcup_{J\subseteq I} \Big[B_0 \cap \Big(\bigcap_{j\in J} \bigcup_{k\in I_j} A_{j,k}\Big) \cap \bigcap_{i\in I-J} A_i^c\Big] \\ &= \bigcup_{\substack{J\subseteq I, \\ (k_j)\in\prod_{j\in J} I_j}} \Big(B_0 \cap \bigcap_{j\in J} A_{j,k_j} \cap \bigcap_{i\in I-J} A_j^c\Big) \\ &= \bigcup_{\substack{J\subseteq I \\ (k_j)\in\prod_{j\in J} I_j}} \Big(B_0 \cap \bigcap_{j\in J} A_{j,k_j} - \bigcup_{i\in I-J} A_i\Big), \end{aligned} \tag{6.11}$$

where the intersection $\cap_{i\in\emptyset}$ of none of sets is assumed to be the whole set. When $J = \emptyset$, then $\prod_{j\in J} I_j$ contains exactly one element (k_j) (the null element), and the intersection $B_0 \cap_{j\in J} A_{j,k_j}$ is just B_0. When $J \neq \emptyset$, if there is at least one $j \in J$ such that J_j is empty, then $\prod_{j\in J} I_j$ contains exactly one element (k_j) (the null element), and the intersection $B_0 \cap_{j\in J} A_{j,k_j}$ is also B_0; if $J_j \neq \emptyset$ for all $j \in J$, since $\mathcal{L}$ is intersectional, then $B_0 \cap_{j\in J} A_{j,k_j}$ is a member of $\mathcal{L}$. Thus the terms in the

union (6.11) are the sets of the form (6.10). Since $\mathcal{B}'$ is closed under finite unions, therefore $B_0 \cap A^c$ belongs to $\mathcal{B}'$. $\square$

Let $\mathscr{L}_i$ be intersectional classes of non-empty sets Ω_i, and $\mathscr{B}_i$ the relative Boolean algebras generated by $\mathscr{L}_i$, $i = 1, 2$. Let $\mathscr{L}_1 \times \mathscr{L}_2$ be the smallest intersectional class of $\Omega_1 \times \Omega_2$ that contains the products $A_1 \times A_2$, where $A_i \in \mathscr{L}_i$; and $\mathscr{B}_1 \times \mathscr{B}_2$ the relative Boolean algebra generated by $\mathscr{L}_1 \times \mathscr{L}_2$. The following lemma can be easily modified to the product of more factors.

Lemma 6.8. *Every member of $\mathscr{B}_1 \times \mathscr{B}_2$ is a finite disjoint union of products $B_1 \times B_2$, where $B_i \in \mathscr{B}_i$.*

Proof. Let $\mathscr{B}$ denote the class of finite disjoint unions of products $B_1 \times B_2$ with $B_i \in \mathscr{B}_i$. It suffices to show that $\mathscr{B}$ is a relative Boolean algebra. Let

$$A = \bigsqcup_{i\in I} A_{1,i} \times A_{2,i}, \quad B = \bigsqcup_{j\in J} B_{1,j} \times B_{2,j},$$

where $A_{1,i} \in \mathscr{B}_1$, $B_{2,j} \in \mathscr{B}_2$. Then

$$A \cap B = \bigsqcup_{i\in I, j\in J} (A_{1,i} \cap B_{1,j}) \times (A_{2,i} \cap B_{2,j}) \in \mathscr{B}.$$

The relative complement $B - A$ can be written as

$$\begin{aligned} B - A &= \bigsqcup_{j\in J}\Big((B_{1,j} \times B_{2,j}) \cap \bigcap_{i\in I}(A_{1,i} \times A_{2,i})^c\Big) \\ &= \bigsqcup_{j\in J}\bigcap_{i\in I}\Big((B_{1,j} \times B_{2,j}) \cap (A_{1,i} \times A_{2,i})^c\Big), \end{aligned}$$

where $(B_{1,j} \times B_{2,j}) \cap (A_{1,i} \times A_{2,i})^c$ can be further written as a disjoint union of three product sets

$$\begin{aligned} A_{i,j,1} &:= (B_{1,j} - A_{1,i}) \times B_{2,j}, \\ A_{i,j,2} &:= B_{1,j} \times (B_{2,j} - A_{2,i}), \\ A_{i,j,3} &:= (B_{1,j} - A_{1,i}) \times (B_{2,j} - A_{2,i}). \end{aligned}$$

Write $[3] := \{1, 2, 3\}$. For each tuple $(k_i) \in \prod_{i\in I}[3]$, it is easy to see that $\cap_{i\in I} A_{i,j,k_i}$ is a product of the form $B_1 \times B_2$, where $B_l \in \mathscr{B}_l$, $l = 1, 2$. Thus

$$B - A = \bigsqcup_{j\in J}\bigcap_{i\in I}\bigsqcup_{k=1}^{3} A_{i,j,k} = \bigsqcup_{j\in J}\ \bigsqcup_{(k_i)\in\prod_{i\in I}[3]}\ \bigcap_{i\in I} A_{i,j,k_i} \in \mathscr{B}.$$

Likewise, $A - B \in \mathscr{B}$. Hence $A \cup B = (B - A) \sqcup (A - B) \sqcup (A \cap B) \in \mathscr{B}$. This means that $\mathscr{B}$ is a relative Boolean algebra. Clearly, $\mathscr{B} \subseteq \mathscr{B}_1 \times \mathscr{B}_2$. Since $\mathscr{B}_1 \times \mathscr{B}_2$ is the smallest relative Boolean algebra containing $\mathscr{L}_1 \times \mathscr{L}_2$, we conclude that $\mathscr{B} = \mathscr{B}_1 \times \mathscr{B}_2$. $\square$

Lemma 6.8 can be easily modified to the product of arbitary finite number of factors of relative Boolean algebras.

Appendix 2: Graph preliminaries

Let $G = (V, E)$ be a graph (loops and multiple edges are allowed). A subgraph H of G is said to be *Eulerian* if the degree of every vertex for H is even. For instance, a connected Eulerian subgraph is just a closed walk without overlapping

edges, called a *closed trail*. A closed simple path is called a *circuit*. For a non-empty proper subset S of V, we denote by $[S, S^c]$ the set of all edges between S and its complement $S^c := E - S$. By a *cut* of G we mean a non-empty edge subset of the form $[S, S^c]$, where $S \subsetneq V$. A *bond* of G is a cut that does not contain properly any cut. Every Eulerian subgraph is an edge-disjoint union of circuits, and every cut is an edge-disjoint union of bonds.

A (combinatorial) *orientation* on G is a (multi-valued) function $\varepsilon : V \times E \to \mathbb{Z}$ such that (i) $\varepsilon(v, e)$ has two values ± 1 if e is a loop at a vertex v and has a single value otherwise, (ii) $\varepsilon(v, e) = 0$ if v is not an end-vertex of e, and (iii) $\varepsilon(u, e)\varepsilon(v, e) = -1$ if u, v are distinct end-vertices of e. Pictorially, an orientation on an edge e can be viewed as that e is equipped with an arrow or direction from its one end-vertex u to the other end-vertex v; such information is encoded by $\varepsilon(u, e) = 1$ and $\varepsilon(v, e) = -1$. So each loop has exactly one orientation combinatorially (however, geometrically or topologically, we may assume that each loop has exactly two orientations); a non-loop edge has exactly two orientations. A graph G with an orientation ε is referred to a *digraph* (G, ε).

A *local direction* of an Eulerian subgraph H is an orientation of H such that the in-degree equals the out-degree at every vertex of H. An Eulerian subgraph may have several local directions. However, if an Eulerian subgraph H is connected and is written as a closed trail W, then there are exactly two local directions on H so that W becomes a directed trail; either of such two local directions on H is called a *direction* of W. Every circuit has exactly two directions. An Eulerian subgraph with a local direction is called a *directed Eulerian subgraph.* Every directed Eulerian subgraph is an edge disjoint union of directed circuits.

A *direction* of a cut $U = [S, S^c]$ is an orientation on U such that the arrows on its edges are all from S to S^c or all from S^c to S. Any cut has exactly two directions; a cut with a direction is called a *directed cut.* A *local direction* of a cut U is an orientation ε_{U} on U such that $(U, \varepsilon_{\mathrm{U}})$ is a disjoint union of directed bonds. A cut may have several local directions.

Let (H_i, ε_i) be directed subgraphs of G, $i = 1, 2$. The *coupling* of ε_1 and ε_2 is a function $[\varepsilon_1, \varepsilon_2] : E \to \mathbb{Z}$, defined for each edge e (at its end-vertex v) by

$$[\varepsilon_1, \varepsilon_2](e) = \begin{cases} 1 & \text{if } e \in E(H_1) \cap E(H_2),\ \varepsilon_1(v, e) = \varepsilon_2(v, e), \\ -1 & \text{if } e \in E(H_1) \cap E(H_2),\ \varepsilon_1(v, e) \neq \varepsilon_2(v, e). \\ 0 & \text{otherwise.} \end{cases} \tag{6.12}$$

Let (G, ε) be a digraph throughout the whole paper. Let A be an abelian group. A function $f : E \to A$ is called a *tension* (or A-*tension*) of (G, ε) if for each directed circuit $(C, \varepsilon_{\mathrm{C}})$,

$$\sum_{e \in C} [\varepsilon, \varepsilon_{\mathrm{C}}](e) f(e) = 0. \tag{6.13}$$

See [**2**] for more details. The set $T(G, \varepsilon; A)$ of all A-tensions forms an abelian group under the obvious addition, called the *tension group* of (G, ε) with values in A. We denote by $T_{\mathrm{nz}}(G, \varepsilon; A)$ the set of all nowhere-zero A-tensions. For each directed bond $(B, \varepsilon_{\mathrm{B}})$, the coupling $[\varepsilon, \varepsilon_{\mathrm{B}}]$ is an integer-valued tension of (G, ε), called a *bond characteristic vector*. It is well-known that the integral tension lattice $T(G, \varepsilon; \mathbb{Z})$ is a $\mathbb{Z}$-span of its bond characteristic vectors.

A function $f : E \to A$ is called a *flow* (or A-*flow*) of (G, ε) if it satisfies the *conservation law:*

$$\sum_{e \in E} \varepsilon(v, e) f(e) = 0 \tag{6.14}$$

at every vertex v of V, where a loop is counted twice at its end-vertex with opposite values. The set $F(G, \varepsilon; A)$ of all A-flows forms an abelian group under the obvious addition, called the *flow group* of (G, ε) with values in A. We denote by $F_{\rm nz}(G, \varepsilon; A)$ the set of all nowhere-zero A-flows. For each directed circuit $(C, \varepsilon_{\rm C})$, the coupling $[\varepsilon, \varepsilon_{\rm C}]$ is an integer-valued flow of (G, ε), called a *circuit characteristic vector*. It is well-known that $F(G, \varepsilon; \mathbb{Z})$ is a $\mathbb{Z}$-span of its circuit characteristic vectors.

Let q be a positive integer. A *real* (*integral*) q-*tension* (q-*flow*) of (G, ε) is a real-valued (integer-valued) tension (flow) f such that $|f(e)| < q$ for all $e \in E$. Let $T_{\rm nz\mathbb{Z}}(G, \varepsilon; q), F_{\rm nz\mathbb{Z}}(G, \varepsilon; q)$ denote the sets of all nowhere-zero integral q-tensions, q-flows of (G, ε), respectively. It is known (see [**1, 8, 14, 19, 20**]) that the counting functions

$$\tau_{\mathbb{Z}}(G, q) := |T_{\rm nz\mathbb{Z}}(G, \varepsilon; q)|, \quad \varphi_{\mathbb{Z}}(G, q) := |F_{\rm nz\mathbb{Z}}(G, \varepsilon; q)| \tag{6.15}$$

are polynomial functions of positive integers q of degree $r(G), n(G)$, called the *integral tension polynomial* and the *integral flow polynomial* of G, respectively, and are independent of the chosen orientation ε, where $r(G)$ is the number of edges of a maximal forest and $n(G)$ is the number of independent circuits of G.

If $|A| = q$ is finite, it is well known that the counting functions

$$\tau(G, q) := |T_{\rm nz}(G, \varepsilon; A)|, \quad \varphi(G, q) := |F_{\rm nz}(G, \varepsilon; A)| \tag{6.16}$$

are polynomial functions of q of degree $r(G), n(G)$, called the *tension polynomial* and the *flow polynomial* of G, respectively, and are independent of the chosen orientation ε and the abelian group structure of A.

Let A^V denote the abelian group of all functions (called *colorations* or *potentials*) from V to A. Recall that a coloration f is said to be *proper* if $f(u) \neq f(v)$ for adjacent vertices u, v. Let $C_{\rm nz}(G, A)$ denote the set of all proper colorations of G. If $|A| = q$ is finite, it is well-known that the counting function

$$\chi(G, q) := |C_{\rm nz}(G, A)| \tag{6.17}$$

is a polynomial function of q, known as the *chromatic polynomial* of G, depending only on the order of A. It is easy to see that

$$\chi(G, t) = t^{c(G)} \tau(G, t), \tag{6.18}$$

where $c(G)$ is the number connected components of G.

There is a *boundary operator* $\partial_\varepsilon : A^E \to A^V$, defined for $f \in A^E$ by

$$(\partial_\varepsilon f)(v) = \sum_{e \in E} \varepsilon(v, e) f(e). \tag{6.19}$$

Then $\ker \partial_\varepsilon$ is the flow group $F(G, \varepsilon; A)$. Potentials and tensions are naturally related by the *co-boundary (*or *difference) operator* $\delta_\varepsilon : A^V \to A^E$, defined for $f \in A^V$ by

$$(\delta_\varepsilon f)(e) = f(u) - f(v), \tag{6.20}$$

where e is an edge whose orientation is from one end-vertex u to the other end-vertex v. Then $\operatorname{im} \delta$ is the tension group $T(G, \varepsilon; A)$.

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List symbols

(G,ε)	a graph $G=(V,E)$ with an orientation ε
B_ε	union of edge sets of directed cuts of (G,ε)
C_ε	union of edge sets of directed Eulerian subgraphs of (G,ε)
$T(G,\varepsilon;A)$	group of tensions of (G,ε) with values in a group A
$T(G,\varepsilon)$	vector space of real-valued tensions of (G,ε)
$T_{\mathbb{Z}}(G,\varepsilon)$	lattice of integer-valued tensions of (G,ε)
$\Delta^+_{\rm TN}(G,B_\varepsilon)$	$\{f\in T(G,\varepsilon):0<f\vert_{B_\varepsilon}<1, f\vert_{C_\varepsilon}=0\}$
$\tau(G,t)$	tension polynomial of G
$\bar\tau(G,t)$	dual tension polynomial
$\tau_{\mathbb{Z}}(G,t)$	integral tension polynomial
$\bar\tau_{\mathbb{Z}}(G,t)$	dual integral tension polynomial
$F(G,\varepsilon;A)$	group of flows of (G,ε) with values in a group A
$F(G,\varepsilon)$	vector space of real-valued flows of (G,ε)
$F_{\mathbb{Z}}(G,\varepsilon)$	lattice of integer-valued flows of (G,ε)
$\Delta^+_{\rm FL}(G,C_\varepsilon)$	$\{f\in F(G,\varepsilon):0<f\vert_{C_\varepsilon}<1, f\vert_{C_\varepsilon}=0\}$
$\varphi(G,t)$	flow polynomial of G
$\bar\varphi(G,t)$	dual flow polynomial
$\varphi_{\mathbb{Z}}(G,t)$	integral flow polynomial
$\bar\varphi_{\mathbb{Z}}(G,t)$	dual integral flow polynomial
$\Omega(G,\varepsilon;A,B)$	tension-flow group $T(G,\varepsilon;A)\times F(G,\varepsilon;B)$
$\Omega(G,\varepsilon)$	tension-flow vector space $T(G,\varepsilon)\times F(G,\varepsilon)$
$\Omega_{\mathbb{Z}}(G,\varepsilon)$	tension-flow lattice $T_{\mathbb{Z}}(G,\varepsilon)\times F_{\mathbb{Z}}(G,\varepsilon)$
$\Omega_{\rm nz}(G,\varepsilon)$	set of nowhere-zero tension-flows in $\Omega(G,\varepsilon)$
$K(G,\varepsilon;A,B)$	$\{(f,g)\in T(G,\varepsilon;A)\times F(G,\varepsilon;B)\mid \operatorname{supp} f=\ker g\}$
$K(G,\varepsilon)$	$\{(f,g)\in T(G,\varepsilon)\times F(G,\varepsilon)\mid f\cdot g=0, f+g\neq 0\}$
$K_{\mathbb{Z}}(G,\varepsilon)$	$K(G,\varepsilon)\cap(\mathbb{Z}^2)^E$
$\Delta_{\rm CTF}(G,\varepsilon)$	$\{(f,g)\in K(G,\varepsilon):0<\vert f+g\vert<1\}$, open 0-1 polyhedron
$\Delta^+_{\rm CTF}(G,\varepsilon)$	$\{(f,g)\in K(G,\varepsilon)\mid 0<f+g<1\}$, open 0-1 polytope
$\kappa_\varepsilon(G;p,q)$	cardinality of dilation $(p,q)\Delta^+_{\rm CTF}(G,\varepsilon)\cap(\mathbb{Z}^2)^E$
$\bar\kappa_\varepsilon(G;p,q)$	cardinality of dilation $(p,q)\bar\Delta^+_{\rm CTF}(G,\varepsilon)\cap(\mathbb{Z}^2)^E$
$\kappa(G;p,q)$	cardinality of $K(G,\varepsilon;\mathbb{Z}_p,\mathbb{Z}_q)$, complementary polynomial
$\bar\kappa(G;p,q)$	dual complementary polynomial
$\kappa_{\mathbb{Z}}(G;p,q)$	cardinality of dilation $(p,q)\Delta_{\rm CTF}(G,\varepsilon)\cap(\mathbb{Z}^2)^E$
$\bar\kappa_{\mathbb{Z}}(G;p,q)$	dual to $\kappa_{\mathbb{Z}}$, dual integral complementary polynomial of G
$R(G;p,q)$	Whitney's rank generating polynomial
$T(G;p,q)$	Tutte polynomial
$\psi(G;p,q,r,s)$	weighted complementary polynomial
$\bar\psi(G;p,q,r,s)$	dual weighted complementary polynomial
$\psi_{\mathbb{Z}}(G;p,q,r,s)$	weighted integral complementary polynomial
$\bar\psi_{\mathbb{Z}}(G;p,q,r,s)$	dual weighted integral complementary polynomial

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